PERISTALTIC FLOW OF A NON-NEWTONIAN LIQUID WITH A POWER-TYPE RHEOLOGICAL LAW IN A SLOT CHANNEL

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The peristaltic motion of a non-Newtonian liquid with a power-type rheological law in a slot channel with elastic, deformable walls is considered. Using the asymptotic method of "narrow bands", the velocity distributions in the channel are derived in terms of certain defining parameters.

An analysis of the flow of liquids through channels with elastic, deformable walls is of particular interest when studying the operating conditions of peristaltic pumps, which are widely employed in the pumping of physiological and structural media [1, 2]. In contrast to ordinary kinds of pumps (centrifugal, pinion, etc.), peristaltic pumps never lead to the separation or disruption of the structure of the liquids being pumped.

In general the peristaltic flow of the medium is caused by the combined action of the deforming channel walls and the pressure gradient applied along the axis. The elastic walls of the working part of the peristaltic pump channel are directly deformed by external action.

In this paper we shall consider the flow of a non-Newtonian liquid with a power-type rheological law [3] in a slot channel, the elastic walls of which are deformed in accordance with the equations of a traveling wave $y = \pm F(x, t)$ having a wavelength much greater than the mean diameter of the channel (Fig. 1). It is obvious that there is a certain reference system, moving in the direction of the channel axis (x) at a velocity W relative to the walls, in which the shape of the upper deformed wall is given (independently of time) by the equation $y = F(x)$; this reference system will in future be called the "moving" system in contrast to the "stationary" system in which the channel walls have no axial displacements. If the flow of liquid in the moving reference system is also independent of time, the corresponding peristaltic motion will be a steady-state process; it is the case with which we shall now be concerned.

The equation of steady-state motion of a non-Newtonian liquid with a power-type rheological law may be written in the form

$$\rho(\vec{v}\nabla)\vec{v} = -\nabla p + \nabla S, \tag{1}$$

$\frac{n-1}{2}$

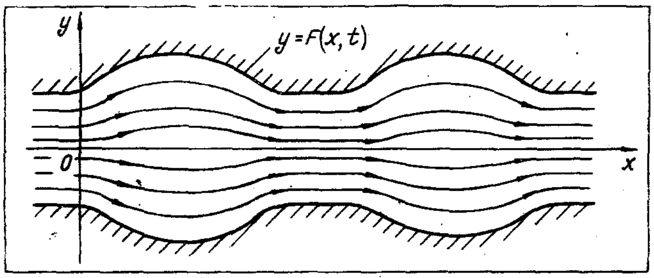

Fig. 1. Picture of the flow of liquid in a channel with elastic deformable walls.

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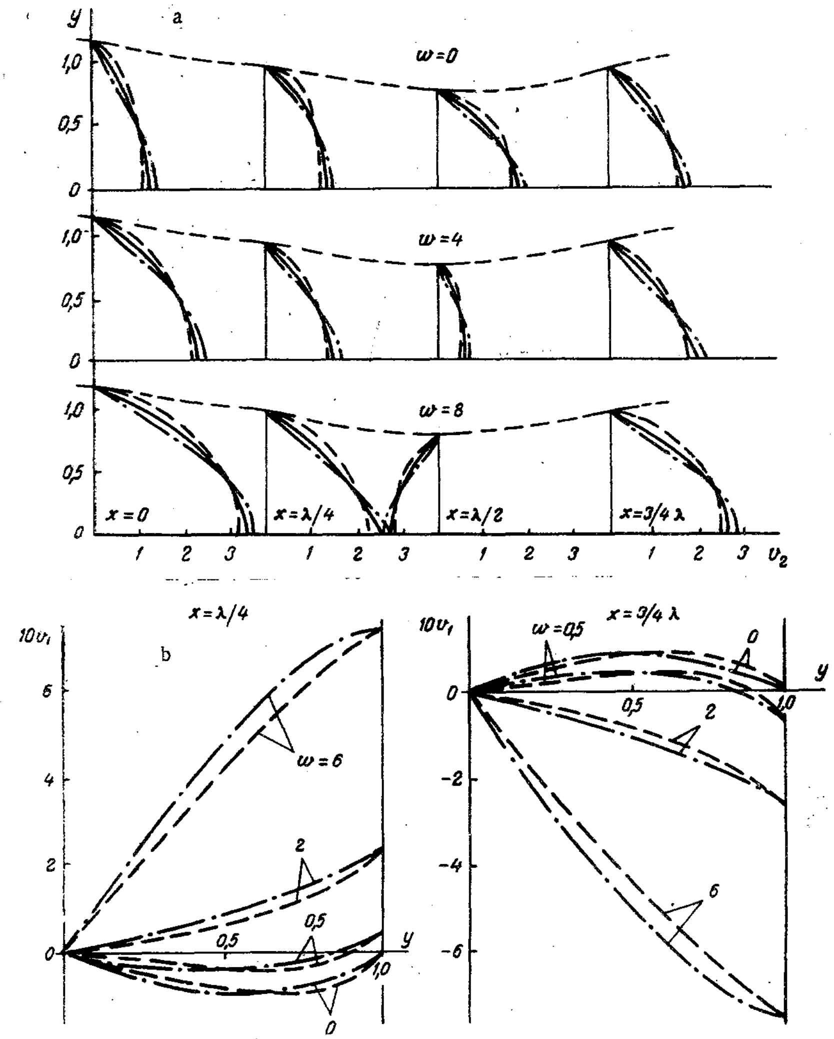

Fig. Distribution of the x projection of the velocity in various cross sections of the channel (a) and the y projection of the velocity in the sections x = λ/4 and x = 3/4 λ (b) for the values n = 0.6 (broken lines), 1.0 (continuous lines), and 1.4 (dotted and dashed lines) and various values of w (the origin x = 0 is placed under the crest of the wave; the calculations are carried out for the values Re_n 100; A = 0.2; λ = 10)

here $\nabla S \equiv \partial s_{ij}/\partial x_j$; $s_{ij} = 2k\,[2f_{ij}^2]^{n-1/2} f_{ij}$, s_{ij} is the deviator of the stress tensor; f_{ij} is the deformation (strain) velocity tensor; k, n are the rheological constants of the medium, and the rest of the nomenclature is obvious [3].

Taking H (the mean half width of the channel) and Q (half the mean rate of flow), as characteristic quantities we may characterize the motion of the liquid by a generalized Reynolds number $Re_n = \rho k^{-1} Q^{2-n} H^{2(n-1)}$ and the dimensionless velocity w = WH/Q. In the moving reference system the equations of motion (1) written in dimensionless form relative to the stream function ψ ($v_1 = -\partial\psi/\partial x$; $v_2 = \partial\psi/\partial y$ are respectively the y and x projections of the velocity) take the following form:

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = -\frac{\partial p}{\partial x} + \frac{\omega^{n-1}}{Re_n}\frac{\partial}{\partial y}\Delta\psi + \frac{n-1}{Re_n}\omega^{n-2}\left[2\frac{\partial\omega}{\partial x}\frac{\partial^2\psi}{\partial x\partial y} + \frac{\partial\omega}{\partial y}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right)\right], \quad (2)$$

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial x\partial y} = -\frac{\partial p}{\partial y} + \frac{\omega^{n-1}}{Re_n}\frac{\partial}{\partial x}\Delta\psi + \frac{n-1}{Re_n}\omega^{n-2}\left[2\frac{\partial\omega}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\omega}{\partial x}\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right)\right]. \quad (3)$$

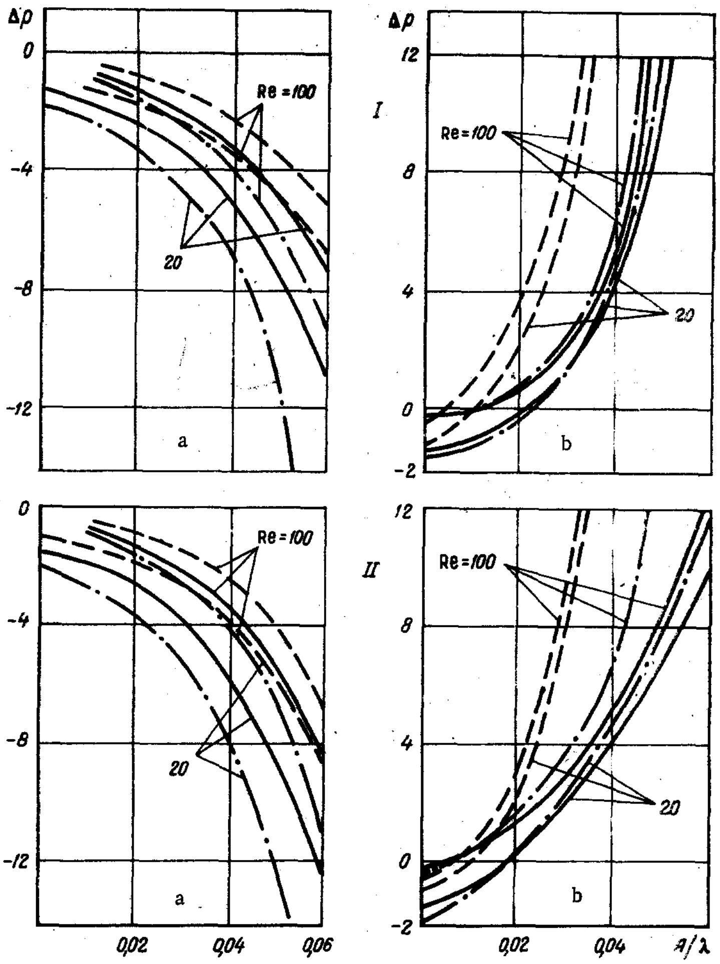

Fig. 3. Change in the pressure drop along the axis between the cross sections: I) x = 0 and x = λ; II) x = λ/2 and x = 3/2 λ for the values a) w = 0; λ = 10; b) w = 4.0; λ = 10. Broken curves n = 0.6; continuous curves n = 1.0; dotted and dashed curves n = 1.4.

Here $\omega \equiv \sqrt{4\left(\frac{\partial^2\psi}{\partial x\partial y}\right)^2 + \left(\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right)^2}$.

If we eliminate the pressure p from Eqs. (2) and (3) we have

$$\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x\partial y^2} + \frac{\partial^3\psi}{\partial x^3}\right) - \frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial y^3} + \frac{\partial^3\psi}{\partial x^2\partial y}\right) = \frac{\omega^{n-1}}{\mathrm{Re}_n}\Delta\Delta\psi + \frac{(n-1)(n-2)}{\mathrm{Re}_n}\omega^{n-3}$$

$$\times\left\{4\frac{\partial\omega}{\partial x}\frac{\partial\omega}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} + \left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right)\left[\left(\frac{\partial\omega}{\partial y}\right)^2 - \left(\frac{\partial\omega}{\partial x}\right)^2\right]\right\}$$

$$+ \frac{n-1}{\mathrm{Re}_n}\omega^{n-2}\left[4\frac{\partial^2\omega}{\partial x\partial y}\frac{\partial^2\psi}{\partial x\partial y} + \left(\frac{\partial^2\omega}{\partial y^2} - \frac{\partial^2\omega}{\partial x^2}\right)\left(\frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial x^2}\right)\right.$$

$$+ \left(\frac{\partial\omega}{\partial y} - \frac{\partial\omega}{\partial x}\right)\left(\frac{\partial^3\psi}{\partial y^3} - \frac{\partial^3\psi}{\partial x^2\partial y}\right) + \left(\frac{\partial\omega}{\partial y}\frac{\partial}{\partial y}\Delta\psi + \frac{\partial\omega}{\partial x}\frac{\partial}{\partial x}\Delta\psi\right)$$

$$\left. + 2\left(\frac{\partial\omega}{\partial x}\frac{\partial^3\psi}{\partial x\partial y^2} + \frac{\partial\omega}{\partial y}\frac{\partial^3\psi}{\partial x^2\partial y}\right)\right]. \quad (4)$$

In the moving reference system, at the upper wall of the channel y = F(x) the stream function ψ should satisfy the conditions

$$\psi = q;\ \frac{\partial\psi}{\partial x} = wF_x;\ \frac{\partial\psi}{\partial y} = -w;\ F_x \equiv \frac{dF}{dx}, \tag{5}$$

where q = 1—w is the dimensionless relative rate of flow of the liquid. The conditions imposed upon the stream function ψ in the center of the channel y = 0 are written in the form

$$\psi = 0;\ \frac{\partial^2\psi}{\partial y^2} = 0. \tag{6}$$

The solution to the problem (4)-(6) may be derived on the basis of the asymptotic method of "narrow bands" [4, 5]. Following the general scheme of this method we make the formal substitution $x \to x/\varepsilon$, where ε is a small parameter characteristic of the "narrowness" of the band (for example, the ratio of the mean half width of the channel to the wavelength of the deformations at the walls of the channel H/Λ); all the foregoing equations and boundary conditions then contain the small parameter ε, so that we may now seek the solution to Eq. (4) in the form of an asymptotic expansion in ε:

$$\psi = \sum_{i=0}^{\infty} \varepsilon^i \psi_i. \tag{7}$$

In order to determine the zero approximation ψ_0 we have:

$$\frac{\partial^2}{\partial y^2}\left(\frac{\partial^2\psi_0}{\partial y^2}\left|\frac{\partial^2\psi_0}{\partial y^2}\right|^{n-1}\right) = 0, \tag{8}$$

which is to be solved with the boundary conditions (5) and (6); in the zero approximation these respectively take the form

$$\psi_0|_{y=F(x)} = q;\ \left.\frac{\partial\psi_0}{\partial x}\right|_{y=F(x)} = wF_x;\ \left.\frac{\partial\psi_0}{\partial y}\right|_{y=F(x)} = -w;\ \psi_0|_{y=0} = 0;\ \left.\frac{\partial^2\psi_0}{\partial y^2}\right|_{y=0} = 0. \tag{9}$$

The function

$$\psi_0 = -\frac{n^2}{(n+1)(n+2)C_1^2}(C_1y + C_2)|C_1y + C_2|^{\frac{n+1}{n}} + C_3y + C_4,$$
$$C_1 = -\left(\frac{2n+1}{nF^{\frac{2n+1}{n}}}\right)^n (q + wF)|q + wF|^{n-1};\ C_2 = 0,$$
$$C_3 = \frac{1}{n+1}\left[wn + (2n+1)\frac{q}{F}\right];\quad C_4 = 0, \tag{10}$$

is the solution of Eq. (8) satisfying the conditions (9). We see from the solution (10) that in those cross sections of the channel x for which $w > [1-F(x)]^{-1} > 0$, the liquid flows in the negative direction of the axis ("counterflow"). We note that the existence of zones of reverse flow for viscous Newtonian liquids was discussed earlier in [6, 7], although no quantitative criteria were derived for the appearance of these zones.

In order to determine the first approximation we have

$$\frac{n(n-1)}{\mathrm{Re}_n}\frac{\partial}{\partial y}\left\{\left|\frac{\partial^2\psi_0}{\partial y^2}\right|^{n-3}\left[\frac{1}{n-1}\left(\frac{\partial^2\psi_0}{\partial y^2}\right)^2\frac{\partial^3\psi_1}{\partial y^3} + \frac{\partial^3\psi_0}{\partial y^3}\frac{\partial^2\psi_0}{\partial y^2}\frac{\partial^2\psi_1}{\partial y^2}\right]\right\} = \frac{\partial\psi_0}{\partial y}\frac{\partial^3\psi_0}{\partial x\partial y^2} - \frac{\partial\psi_0}{\partial x}\frac{\partial^3\psi_0}{\partial y^3}, \tag{11}$$
$$\psi_1|_{y=F(x)} = 0;\ \left.\frac{\partial\psi_1}{\partial x}\right|_{y=F(x)} = 0;\ \left.\frac{\partial\psi_1}{\partial y}\right|_{y=F(x)} = 0;\ \psi_1|_{y=0} = 0;\ \left.\frac{\partial^2\psi_1}{\partial y^2}\right|_{y=0} = 0,$$

whence

$$\psi_1 = \Omega_1 C_5 + \Omega_2 C_6 + C_7 y + C_8 + \Omega_3. \tag{12}$$

Here

$$\Omega_1 = \frac{n\,\mathrm{Re}_n}{(n+1)(2n+1)}|C_1|^{\frac{1-n}{n}} y^{\frac{2n+1}{n}};\quad \Omega_2 = \frac{n\,\mathrm{Re}_n}{n+1}|C_1|^{\frac{1-n}{n}} y^{\frac{n+1}{n}};$$

$$\Omega_3 = \frac{n^2\mathrm{Re}_n}{2(n+1)^2(2n+1)(3n+2)}|C_1|^{\frac{2(1-n)}{n}} y^{\frac{3n+2}{n}}\left[|C_1|\dot{C}_3 - C_3|\dot{C}_1| + \frac{2n^2}{3(n+1)(4n+3)}|C_1|^{\frac{1}{n}}|\dot{C}_1| y^{\frac{n+1}{n}}\right];$$

$$C_5 = \frac{n}{(n+1)(3n+2)}|C_1|^{\frac{1-n}{n}} F^{\frac{n+1}{n}}\left[C_3|\dot{C}_1| - \dot{C}_3|C_1| - \frac{n^2}{(n+1)(4n+3)}|C_1|^{\frac{1}{n}}|\dot{C}_1|F^{\frac{n+1}{n}}\right];\quad C_6 = 0;$$

$$C_7 = \frac{n^2\,\mathrm{Re}_n}{2(n+1)^2(2n+1)(3n+2)}|C_1|^{\frac{2(1-n)}{n}} F^{\frac{2(n+1)}{n}}\Big[|C_1|\dot{C}_3 - C_3|\dot{C}_1|$$

$$+\frac{4n^2}{3(n+1)(4n+3)}|C_1|^{\frac{1}{n}}|\dot{C}_1|F^{\frac{n+1}{n}}\Big];\quad C_8 = 0;\ \dot{C}_i = \frac{dC_i}{dx}\quad (i = 1;\ 3).$$

We have thus constructed an asymptotic solution for the stream function ψ in the moving reference system, to an accuracy limited by terms of the order $O(\varepsilon^2)$. Applying this solution to the Minsk—2 computer we calculated the velocity profiles and pressure drops along the axis of the channel for the particular case in which the deformation of the elastic walls of the channel in the "stationary" reference system was described by the law of a monochromatic wave

$$y = F(x,\ t) \equiv 1 + A\sin 2\pi\left(\frac{t}{T} - \frac{x}{\lambda}\right)\quad (0 < A < 1), \tag{13}$$

where A, T, $\lambda = \Lambda/H$ are respectively the amplitude, period, and dimensionless wavelength.

Figure 2a and b illustrates the distribution of the x and y velocity projections in relation to the defining parameters of the problem, the case w = 0 corresponding to flow in a corrugated channel.

The pressure distribution inside the channel may be determined if the asymptotic expression constructed for the function ψ is substituted into Eqs. (2) and (3).

Figure 3, I illustrates the pressure drop along the axis of the channel between the sections x = 0 and x = λ (Fig. 2a) in relation to the ratio A/λ. Whereas for a pressure drop $\Delta p = 0$ the motion of the liquid takes place solely on account of the deformation of the elastic walls of the channel, for $\Delta p \neq 0$ the motion takes place by virtue of the action of the pressure gradient as well as the deformation of the walls; for $\Delta p > 0$ these effects oppose one another.

Figure 3,II illustrates the pressure drop along the axis of the channel between the sections x = $\lambda/2$, x = 3/2 λ (Fig. 2b).

NOTATION

x and y	are the Cartesian coordinates;
W	is the phase velocity of the wave along the channel walls;
F	is the function describing the shape of the deformed wall;
ρ	is the density of the medium;
v	is the velocity vector;
p	is the pressure;
s_{ij}	is the deviator of the stress tensor;
f_{ij}	is the deformation rate tensor;
n, k	are the rheological constants;
Re_n	is the generalized Reynolds number;
Q	is half the mean flow rate;
H	is the half width of channel;
ψ	is the stream function;
q	is the dimensionless relative flow rate;
w	is the dimensionless phase velocity;
ε	is the small parameter;
Λ	is the wavelength of the deformations at the walls;
λ	is the dimensionless wavelength;
$C_1, C_2, C_3, \Omega_1, \Omega_2, \Omega_3, \ldots$	are unknown functions;
A	is the amplitude of the wave;
T	is the period of the wave.

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